Lesson 10a – Problem-Solving Strategy

AIM: How do we identify and apply problem-solving strategies?

PERFORMANCE STANDARDS: M1, M5, M7

PERFORMANCE OBJECTIVES:

1. Define problem solving.
2. Identify and apply most commonly used problem-solving techniques in solving a task:
   Working backwards, finding a pattern, solving a simpler analogous problem. A visual representation, intelligent guessing and checking, organizing data logical reasoning, accounting for all possibilities, considering extreme cases, and adopting a different point of view.

DO NOW:

1. How do we evaluate algebraic expressions using given values from the set of signed numbers?
2. Using the rule you just wrote evaluate:
   \[ x^2 + y (2w - x)^2 \] when \( x = -1, \ y = -2. \)

MOTIVATION:

Compare and contrast problem-solving strategies to taking different modes of transportation to your friend’s house.

DEVELOPMENT:

Refer to the motivation and elicit that since there are several ways to get from one place to the same location, there are also different strategies or methods we can use to solve a problem.

Elicit a definition of problem-solving as a way of:
1. Understanding the problem by reading and asking yourself “What information is given? What am I looking for?”
2. Making a plan by organizing the necessary information and deciding upon a strategy.
3. Solving the problem by carrying out the steps in the plan.
4. Checking the solution to see if it makes sense and satisfies the conditions in the problem.

Elicit and write problem-solving strategies that the students have used. Distribute instructional strategies. (Make a poster of the instructional strategies for your classroom).

For the remaining part of the lesson and throughout the Math A and Math B program incorporate real-life problem-solving tasks. Encourage the students to use a variety of strategies.

**INSTRUCTIONAL STRATEGIES:**

- Working backwards
- Finding a pattern
- Adopting a different point of view
- Solving a simpler analogous problem
- Considering extreme cases
- Making a visual representation
- Intelligent guessing and testing
- Accounting for all possibilities
- Organizing date
- Logical reasoning

**PROBLEM-SOLVING STRATEGIES:**


**WORKING BACKWARD:**

This strategy is useful when you know the result of a series of events and want to find a value present at the beginning of the series. Inverses are often used when working backward. If a problem involves addition or subtraction, use the additive or opposite. If a problem involves multiplication or division, use the multiplicative inverse or reciprocal.
EXAMPLES:

1. Bob runs the elevator in an apartment building. He took Mr. Sloan up six floors from the floor on which he lives. Then Bob went down five floors, where he picked up Mrs. Rice. He took her down ten floors to the first-floor lobby. What is the number of the floor on which Mr. Sloan lives?

**Plan:** Work backward, reversing the course of the elevator, to find the number of Mr. Sloan’s floor.

**SOLUTION:** From the lobby, go up 10 floors to where Mrs. Rice got on. This is the 11th floor. Go up 5 more floors to where Mr. Sloan got off. This is the 16th floor, go down 6 floors to where Mr. Sloan got on. This is the 10th floor.

**ALTERNATIVE:**

The elevator went up 6 floors, down 5 floors, then down 10 to get to floor 1.

**SOLUTION:** Use integers: +6 -5 and -10 to get 1

- Start at 1 and apply additive inverses in the reverse order of the problem.
- The additive inverse of -10 is +10, so: 1 + 10 = 11
- The additive inverse of -5 is +5, so: 11 + 5 = 16
- The additive inverse of +6 is -6, so 16 - 6 = 10 (the 10th floor).

**Check:** Let Mr. Sloan get on at the 10th floor and follow the original course of the elevator. The elevator went from the 10th floor up 6 floors to the 16th. From there, it went down 5 floors, to the 11th, and then down the remaining 10 floors to the 1st floor, or lobby.

**Answer:** Mr. Sloan lives on the 10th floor.

2. Each weekend Jamaal earns money, delivering groceries. He deposits four-fifths of what he earns in his savings account and keeps the rest to spend. Last weekend, he kept $12. How much did Jamaal earn?
Plan: If Jamaal deposited $\frac{4}{5}$ of his money, he kept $\frac{1}{5}$ to spend.

Therefore, $12$ is $\frac{1}{5}$ of what he earned.

SOLUTION: When a fractional part of an amount is calculated, multiplication is the operation used. Since $12$ is $\frac{1}{5}$ of Jamaal’s earnings, multiply by the inverse, $\frac{5}{1}$ of what he earned.

$12 \times 5 = 60$

He earned $60$.

Check: $\frac{4}{5} \times 60 = 48$. Therefore, $48$ was saved.

$60 - 48 = 12$ therefore, $12$ was left to spend

Answer: Jamaal earned $60$ last weekend.

DISCOVERING PATTERNS: Glencoe page 82 introduction Ex.1 and 2.

DISCOVERING PATTERNS:
Many problems about sets of numbers that follow a pattern can be solved by making use of the patterns involved.

EXAMPLE:
1. What is the next number in the following sequence?
   1, 2, 4, 11,…

   Plan: Since the numbers are increasing whole numbers, look for patterns that add whole numbers or multiply by whole numbers.

   SOLUTION: Look for a multiplication pattern. The second number is twice the first, and the third number is twice the second. However, the fourth number is not twice the third. Thus, the sequence is not the result of multiplication by the same number. Look for an addition pattern.

   \[
   \begin{array}{cccc}
   1 & 2 & 4 & 7 \\
   +1 & +2 & +3 & +4 \\
   \end{array}
   \]

   Each number is obtained by adding a number larger by 1 than the number previously added. If this pattern is continued, the next number is obtained by adding 5 to 11. Using this pattern, you find the next number is 16.
Check:  
\[ 1 \quad 2 \quad 4 \quad 7 \quad 11 \quad 16 \]
\[ +1 \quad +2 \quad +3 \quad +4 \quad +5 \]

Answer: The next number in the sequence is 16.

2. What is the sum of the whole numbers 1 through 80?

**Plan:** To estimate an answer, consider that, of these 80 numbers, the middle, or “average,” number is about 40. If 80 numbers have an average of 40, then \(80 \times 40\) or 3,200 is as good estimate of the sum of these numbers.

One possible solution is simply to add the 80 numbers or you can look for a pattern. There are many ways to find a pattern. Two methods are shown in the solutions that follow.

**SOLUTION:** Match the smallest and largest numbers. Then match the second smallest with the second largest.

Notice that \(1 + 80 = 81\), \(2 + 79 = 81\), and so on.

\[
\begin{align*}
1 & + 2 \quad + 3 \quad + 4 \quad + \ldots + 77 \quad + 78 \quad + 79 \quad + 80 \\
\end{align*}
\]

Since there are 80 numbers, there are \(\frac{1}{2} \times 80\) or 40 pairs. The sum of these 40 pairs is \(40 \times 81\) or 3,240.

**Alternative Solution:** Write the sum twice: First with the numbers in increasing order as given, and then, on a second line, in decreasing order. Add the two lines as shown below:

\[
\begin{align*}
1 & + 2 \quad + 3 \quad + 4 \quad + 5 \quad + \ldots + 76 \quad + 77 \quad + 78 \quad + 79 \quad + 80 \\
80 \quad + 79 \quad + 78 \quad + 77 \quad + 76 \quad + \ldots + \frac{5}{2} \quad + \frac{4}{2} \quad + \frac{3}{2} \quad + \frac{2}{2} \quad + \frac{1}{2} \\
81 \quad + 81 \quad + 81 \quad + 81 \quad + 81 \quad + \ldots + 81 \quad + 81 \quad + 81 \quad + 81 \quad + 81 \\
\end{align*}
\]
There are 80 pairs, each having a sum of 81. This third line, which equals 80(81), is twice the sum of the whole numbers from 1 to 80. Therefore, the sum we are looking for is one-half of 80(81) or

\[ \frac{80(81)}{2} = 40(81) = 3,200 \]

**Check:** You could use a calculator to check this sum. Notice that the result is close to the estimate.

**Answer:** The sum of the whole numbers 1 through 80 is 3,200.

**SOLVING A SIMPLER ANALOGOUS PROBLEM:**

**USING A SIMPLER RELATED PROBLEM:**
If a problem uses large numbers or consists of many cases, it is often possible to find the solution by first solving a similar problem with smaller numbers or fewer cases.

Study the following examples that are solved by using a simpler related problem. In Example 1, the problem is changed from finding the sum of the whole numbers from 30 to 39 to finding the sum of the whole numbers from 0 to 9. In Example 2., the problem is changed from forming, under given conditions, three stacks of pennies made with 100 pennies to finding the number of pennies needed for the smallest possible stacks with the given conditions. In each case, the simpler problem leads to a solution of the original problem.

**INTEGRATED MATHEMATICS COURSE I (page 78).**

**EXAMPLE:**
A man has 100 pennies. He tries to place them all in three stacks so that the second stack has twice as many pennies as the first, and the third stack has twice as many pennies as the second.

a) What is the largest number of pennies that can be placed in each stack?
b) How many pennies are left over?

**Plan:** Make the smallest possible stacks, and determine how many such stacks can be made with 100 pennies.

**SOLUTION:** a) and b). To make the smallest possible stacks (1, 2, and 4 pennies), you need 7 pennies. To find how many such stacks can be made from 100 pennies, divide:

\[ 100 \div 7 = 14, \text{ with a remainder of } 2 \]

Therefore, you can make each of these small stacks 14 times as large and use all but two of the pennies.

\[
\begin{align*}
14(1) &= 14 \text{ first stack} \\
14(2) &= 28 \text{ second stack} \\
14(4) &= 56 \text{ third stack} \\
&\quad 98 \text{ pennies.}
\end{align*}
\]

There would be 2 pennies left over.

**ALTERNATIVE SOLUTION:** Start with simple numbers that fit the problem. Then use multiplication to increase these numbers until you find the answer.

<table>
<thead>
<tr>
<th>Stack 1</th>
<th>Stack 2</th>
<th>Stack 3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>The smallest possible Numbers are</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Multiply each of these Numbers by 10:</td>
<td>10</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>Multiply the top row by 15:</td>
<td>15</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>Multiply the top row by 14:</td>
<td>14</td>
<td>28</td>
<td>56</td>
</tr>
</tbody>
</table>

There would be \(100 - 98\) or 2 pennies left over.

**Check:** The second stack has twice as many as the first stack \((2 \cdot 14 = 28)\), and the third stack has twice as many as the second stack \((2 \cdot 28 = 56)\).
**Answers:**  
a. The three stacks contain 14, 28, and 56 pennies.  
b. Two pennies are left over

**MAKING A VISUAL REPRESENTATION:**  
Drawing pictures and diagrams, Amsco, pages 84 and 85  
Introduction Ex. 1.

**DRAWING PICTURES AND DIAGRAM:**  
A picture or diagram can often help you visualize a problem and may suggest a way to solve it.

**EXAMPLE:**

Mr. Vroman had a rectangular vegetable garden. He decided to increase the size by making the length twice that of the original garden and the width 3 times that of the original garden. How many times as large as the original garden is the new garden.

**Plan:** Draw a diagram of the original garden and of the new garden and compare them.

**SOLUTION:** Sketch the original garden.

```
  ____________
```

Double the length.

```
  ____________
  |          |
  |          |
  |          |
  |          |
  |__________|
```

Triple the width.

```
  ____________
  |          |
  |          |
  |          |
  |          |
  |          |
  |__________|
  |          |
  |          |
  |__________|
```

The new garden consists of six gardens each of the same size as the original one.

**Problem Solving:**

**Check:** Assign numbers to the length and width to show that the relationship holds.

**Original:**
- 1 by 5  Area = 5
- 3 by 4  Area = 12

**New:**
- 3 by 10  Area = 30
- 9 by 8  Area = 72

\[
\begin{align*}
30 &= 6(5) \\
72 &= 6(12)
\end{align*}
\]

**Answer:** The new garden is 6 times as large as the original one.

**INTELLIGENT GUESSING AND TESTING:**

Guessing and checking, Amsco, pages 75 and 76 Introduction Ex. 1.

**GUESSING AND CHECKING:**

This strategy is often called trial and error. It is particularly useful when the answer must be a whole number between limits.

**EXAMPLES**

1. Keegan celebrated his birthday by inviting a group of friends to join him in having sundaes. The bill amounted to $27.43. If the cost of each sundae was the same: a) How many friends did Keegan invite? b) What was the cost of a sundae?

**Understand:** Think of the amount of the bill as 2,743 cents. The cost, in cents. Of each sundae and the number of sundaes must be whole numbers whose product is 2,743. If the product of two whole numbers is 2,743, then 2,743 divided by one of the whole numbers is the other whole number.

**Plan:** Guess a possible number of sundaes. Use a calculator to divide 2,743 by the number. If the quotient is a whole number, the guess and the quotient are possible solutions. If the quotient is not a whole number, guess another number. Since 2,743 is an odd number. It must be the product of two odd numbers.
Also, neither of these odd numbers can end in 5 because the units digits of 2,743 is not 5 or 0.

**SOLUTIONS:** a) and b) Divide 2,743 by odd numbers that do not end in 5, (that is, divide by 3, 7, 9, 11, 13, 17,...) until a whole-number quotient is found.

Enter:

2,743 ÷ 3 =

Display: 914.333333

No

Enter:

2,743 ÷ 7 =

Display: 391.85714

No

Enter:

2743 ÷ 9 =

Display: 304.77778

No

Enter:

2743 ÷ 11 =

Display: 211.

No

In each case shown above, the quotient is not a whole number.

**INTEGRATED MATHEMATICS: COURSE I (page 76).**

Enter:

2743 ÷ 13 =

Display: 211.

A whole number

Since 13(211) = 2,743:

or 13 sundaes at $2.11 each = $27.43 (reasonable answers)

211 sundaes at $0.13 each = $27.43 (211 is not reasonable number of people, nor is $0.13 a reasonable cost for a sundae).

Are these the only possible answer? If we try other numbers, we find that 13 and 211 are the only two whole numbers, other than 1 and 2,743, whole product is 2,743. No solution makes sense except the one found.

**Check:** 13($2.11) = $27.43
**Answers:** a. Keegan invited 12 friends (the 13 sundae was for himself).
b. Each sundae cost $2.11.

**ORGANIZING DATA:**
Making lists and charts. Amsco, page 87 Introduction, Pages 87-88 Ex. 1.

**MAKING LIST AND CHARTS:**
Problem in which different possible solutions are to be investigated can often be solved by making organized lists or charts.

**EXAMPLES:**
1. Three friends, Jane, Rose, and Phyllis, study different languages and have different career goals. One wants to be an artist, one a doctor, and the third a lawyer.
   1) The girl who studies Italian does not plan to be a lawyer.
   2) Jane studies French and does not plan to be an artist.
   3) The girl who studies Spanish plans to be a doctor.
   4) Phyllis does not study Italian.
Find the language and career goal of each girl.
Plan: Make a chart and fill in the information, starting with the most definite clues first.

**SOLUTION:**

<table>
<thead>
<tr>
<th></th>
<th>Jane</th>
<th>Rose</th>
<th>Phyllis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Career</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Clue (2) gives the most definite information about language. After using that use clue (4) to determine the other languages.

<table>
<thead>
<tr>
<th></th>
<th>Jane</th>
<th>Rose</th>
<th>Phyllis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language</td>
<td>French</td>
<td>Italian</td>
<td>Spanish</td>
</tr>
<tr>
<td>Career</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Now that you know which girl studies which language, Clue (3) gives definite information about a career choice. Clues (1) and (2) enable you to determine the other career choices.

<table>
<thead>
<tr>
<th></th>
<th>Jane</th>
<th>Rose</th>
<th>Phyllis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Language</strong></td>
<td>French</td>
<td>Italian</td>
<td>Spanish</td>
</tr>
<tr>
<td><strong>Career</strong></td>
<td>lawyer</td>
<td>artist</td>
<td>doctor</td>
</tr>
</tbody>
</table>

INTEGRATED MATHEMATICS: COURSE I:

**Check:** Compare each clue with the chart.
Rose studies Italian and does not plan to be a lawyer.
Jane studies French and does not plan to be an artist.
Phyllis studies Spanish and plans to be a doctor.
Phyllis does not study Italian.

**Answer:** The chart displays the language and career goal of each girl.

ADOPTING A DIFFERENT POINT OF VIEW: Can put things into focus. For example, if you want to cut a two layer circular cake into 8 equal pieces with only 3 straight cuts, you might try to cut it straight down realizing that you end up with six slices. If you look at the cake a different way you can make two vertical cuts and one horizontal cut and end up with 8 equal slices.

CONSIDERING EXTREME CASES: Can be useful in making a problem more understandable. Try a range of values to get a handle on the situation, even if the extreme cases are not realistic. For example, A V.C.R sales manager claimed that 87% of the V.C.R’s sold during the last 8 years were still operating. Does her statement mean that her V.C.R’s have an 87% chance of lasting 8 years?
If you consider the extreme cases that most of the V.C.R’s were sold 8 years ago, then they probably have less than an 87% chance of lasting 8 years. However, If most of them were sold within the last year, then it would indicate an 87% chance of lasting 8 years. More data is needed.

ACCOUNTING FOR ALL POSSIBILITIES: Especially in life, can prevent things from going wrong. List all of the possibilities and exam it for repetition. For example: Ms Tomosullo is choreographing a fashion and talent show at her High school. There are 50 performances who can sing, dance, model and/or act. How can she arrange the performances in the show so that the students will have enough time to change between their performances and the stage crew has enough time to change the scenery? Ms Tomosullo must account for every person by: Making a list so she doesn’t leave out anyone, rearranging the performances; and crossing out duplication.

LOGICAL REASONING: Can be used to help you figure things out. A detective uses logical reasoning to solve mysteries. Refer to the problem from making lists and charts. Logical reasoning helps you to identify possible answers.

There are other strategies to solving a problem like setting up an equation and solving it. This we will learn about in a few days. In order to be a problem-solver we must be able to keep trying, take chances, use what you know practice, watch you do, you might want to use a different strategy.

SUMMARY:

1. What should you do when you are presented with a problem?
2. What are some of the most commonly used problem solving strategies.
3. How can lists or diagrams help you to solve a problem.
**HOMEWORK:** Spiraled.

**Math A Regents - June 1999.**

A swimmer plans to swim at least 100 laps during a 6-days period. During this period, the swimmer will increase the number of laps completed each day by one lap. What is the least number of laps the swimmer must complete on the first day?

**Method I: Trial and Error**

<table>
<thead>
<tr>
<th>Day</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>First day</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>75</td>
</tr>
<tr>
<td>Second day</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>87</td>
</tr>
<tr>
<td>Third day</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>99</td>
</tr>
<tr>
<td>Fourth</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>105</td>
</tr>
<tr>
<td>Fifth day</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>111</td>
</tr>
<tr>
<td>Sixth day</td>
<td>15</td>
<td>17</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>117</td>
</tr>
</tbody>
</table>

Start with any amount of laps for the first day and increase the number of laps by 1 for the next days. Take the total. For example, the table above starts with 10 laps for the first day, the total is 75. Starting with 14 laps on the first day will yield 99 total laps, which is 1 shy of 100. Starting with 15 laps on the first day will yield 105 laps, this is the least number of laps the swimmer must complete on the first day.

**Method II: Algebraic Method**

Let $x = \#$ of laps completed by the swimmer on the first day

$x + 1 = \#$ of laps completed by the swimmer on the second day

$x + 2 = \#$ of laps completed by the swimmer on the third day

$x + 3 = \#$ of laps completed by the swimmer on the fourth day

$x + 4 = \#$ of laps completed by the swimmer on the fifth day

$x + 5 = \#$ of laps completed by the swimmer on the sixth day

$x + x + 1 + x + 2 + x + 3 + x + 4 + x + 5 \geq 100$

$6x + 15 \geq 100$ (Put the “like” terms together)
\[6x \geq 85 \text{ (Subtract 15 from both sides)}\]

\[x \geq 14.1666667\]

\[x = 15 \text{ laps must be completed by the swimmer on the first day}\]

(15 is the closest integer greater than 14.1666667)